

On Some Exponential Potentials for a Cosmological Scalar Field as Quintessence

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Abstract

We present general exact solutions for two classes of exponential potentials in a scalar field model for quintessence. The coupling is minimal and we consider only dust and scalar field. To some extent, it is possible to reproduce experimental results from supernovae.

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I. INTRODUCTION

In the last few years, new models of the universe have been built taking *dark energy* into account [1,2]. Together with baryons, cold dark matter, photons and neutrinos, a fifth component has been added, the so-called *quintessence* field Q [3–7] (or, in general, the *x-field* [8,9]). Generalizing ideas like that of a cosmic equation of state variable with a Λ -term [10], with respect to a more usual cosmological constant Λ , such a Q -field, even if still implying a negative pressure contribution to the total pressure of the cosmic fluid, is characterized by the fact that its equation of state is given by $-1 < w_Q \equiv p_Q/\rho_Q < 0$, p_Q and ρ_Q being, respectively, the pressure and energy density of the Q -field. Actually, the interval $-1 < w_Q \lesssim -0.6$ is usually considered [11]. As a matter of fact, when $w_Q = -1$ we recover a constant Λ -term [12–14], which can be regarded as a measure of vacuum energy density, leading to the well known discrepancy between theory and observations [15,12,13], based on the question of why ρ_Q is so small with respect to typical particle physics scales. (But there

are also mechanisms of relaxation of the cosmological constant during the initial inflationary stage, which could explain such a discrepancy; see [16], for instance.)

As well known, an interesting possibility to handle the presence of quintessence in the universe is to see it as given by a scalar field φ slowly rolling down its potential $V(\varphi)$. If we define

$$\rho_\varphi \equiv \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \quad ; \quad p_\varphi \equiv \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \quad (1)$$

(dot indicating time derivative, and $V(\varphi)$ being the potential for φ), the slow rolling condition immediately gives $p_\varphi < 0$ and $w_\varphi \equiv p_\varphi/\rho_\varphi \simeq -1$. With such a negative pressure, the universe evolves like in a sort of present day *soft* inflationary scenario, so allowing to explain observations on supernovae [17–21] and why vacuum and matter densities are today comparable (‘cosmic coincidence’ problem [22,12,13]). (Alternatively to quintessence, a negative pressure and an explanation of current observations can also be obtained in a *Chaplygin cosmology* [23].)

Many cosmological models with a dynamical scalar field have been proposed, showing *scaling* solutions, i.e., such that at some time ρ_m and ρ_φ simultaneously depend on some powers n_1 and n_2 of the scale factor a , acting as attractors in the phase space. When $n_1 = n_2$, we have the so-called *self-tuning* solutions [24], which are typically driven by exponential potentials. This kind of potential has been studied extensively [25–34], especially from a qualitative point of view (see [35] and references therein, for instance).

The simplest possibility is of course $V(\varphi) = \alpha e^{\lambda\varphi}$, which is often discarded (see discussion in [6,14]). More promising seems to be a combination of two terms $V(\varphi) = \alpha e^{\lambda\varphi} + \beta e^{-\lambda\varphi}$ [32].

In this paper, we consider a particular class of both these types from a different point of view, obtaining *general exact solutions*. This allows a very stringent comparison with experimental data on supernovae, so that also the first type seems to deserve further investigations; for the second type, we obtain a solution which can mimic very well the presence of a cosmological constant in the late evolution of the universe. Both of them are not, strictly speaking, *scaling solutions*, although this concept may be recovered in a more general sense.

Another experimental fact which we use is the strong evidence of a spatially flat universe [36]. Thus, we set the scalar curvature $k = 0$ in all our equations from the very beginning. However, as we shall see, the values of Ω_m and Ω_Λ (Ω_φ in our case) derived from the experiments strongly depend on the model, so that some discussion is needed.

Mostly, the scalar field φ has been considered as minimally coupled to gravity, even if (more recently) a nonminimal coupling has also been introduced [37–42]. Here, we will consider a very simple model consisting of a two-component cosmological fluid: matter and scalar field. ‘Matter’ means baryonic + cold dark matter, with no pressure, and the scalar field is minimally coupled and noninteracting with matter. Clearly, this model cannot be used from the very beginning of the universe, but only since decoupling of radiation and dust. Thus, we do not take into account inflation, creation of matter, nucleosynthesis, etc. The main shortcut is that we cannot really check for the *tracker* feature [5,6] of the φ -solution.

In Sec. II we take a particular exponential potential into account, deriving the general exact solution of the cosmological equations and, thereby, cosmological parameters as functions of time, so allowing the comparison with observational data. Sec. III is devoted to

the same kind of considerations for a potential given by a linear combination of two such exponentials. Conclusions are drawn in Sec. IV.

II. AN EXPONENTIAL POTENTIAL

A. Mathematical treatment

Let us consider a spatially flat, homogeneous and isotropic universe, filled with two noninteracting components only, i.e., pressureless matter (or *dust*) and a scalar field φ , minimally coupled to gravity. The cosmological equations are then

$$3H^2 = \frac{8\pi G}{c^2}(\rho_m + \rho_\varphi), \quad (2)$$

$$\dot{H} + H^2 = -\frac{4\pi G}{3c^2}(\rho_m + \rho_\varphi + 3(p_m + p_\varphi)), \quad (3)$$

$$\ddot{\varphi} + 3H\dot{\varphi} + V'(\varphi) = 0, \quad (4)$$

where prime indicates derivative with respect to φ , $H \equiv \dot{a}/a$ is the Hubble parameter, $p_m = w_m \rho_m$ and $p_\varphi = w_\varphi \rho_\varphi$ are the equations of state for matter and scalar field. Let us stress that w_φ is *not constant*, and that we want to describe some features of cosmology after the decoupling.

We set $w_m = 0$, so that $\rho_m = Da^{-3}$. The parameter $D \equiv \rho_{m0}a_0^3$ (the lower index ‘0’ indicating present day values) is the amount of matter. The equations can also be rewritten as

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3c^2}(Da^{-3} + \frac{1}{2}\dot{\varphi}^2 + V(\varphi)), \quad (5)$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 = -\frac{8\pi G}{3c^2}\left(\frac{1}{2}\dot{\varphi}^2 - V(\varphi)\right), \quad (6)$$

$$\ddot{\varphi} + 3\left(\frac{\dot{a}}{a}\right)\dot{\varphi} + V'(\varphi) = 0. \quad (7)$$

In this Section, we consider the potential

$$V(\varphi) = B^2 e^{-\sigma\varphi}, \quad (8)$$

where B^2 is a generic positive constant and

$$\sigma^2 \equiv \frac{12\pi G}{c^2}. \quad (9)$$

(The minus sign in the exponential is irrelevant, since there is symmetry with respect to a change $\varphi \rightarrow -\varphi$.)

This type of potential leads to a late time attractor in a scalar-field dominated situation ($\Omega_\varphi = 1, w_\varphi = -0.5$) [14,32]. Being aware of such a behaviour, anyway, we stress that we are especially interested in the contemporary or, at most, the recent past regimes, where the situation is different. Usually, associated with an exponential potential, a scalar field

is considered such that $\Omega_\varphi \equiv 8\pi G\rho_\varphi/(3c^2H^2)$ is practically constant during part of the matter-dominated era. This implies that assuming $w_\varphi \sim \text{constant}$ leads to a constant ratio of quintessence to matter energy density, so that Ω_φ (being $\lesssim 0.15$ at the beginning of matter-dominated era, due to nucleosynthesis [29,30]) must remain small forever [6,14]. Mainly for such reasons, this kind of potential is not considered as suitable for a quintessence field.

The particular choice of Eq. (9) for σ allows for general exact integration of equations. Such a choice was in fact used in the context of inflationary theory by us [43,44] and others [45,46], with a scalar field only.

Let us concentrate on the second order equations (6) and (7), while Eq. (5), which is a first integral, is considered as a constraint on the integration constants. Let us introduce the new variables u and v , defined by the transformation

$$a^3 = uv \quad ; \quad \varphi = -\frac{1}{\sigma} \log \frac{u}{v}, \quad (10)$$

which is always invertible (the Jacobian being $J = 2/\sigma$). We get for the potential

$$V(u, v) = B^2 \frac{u}{v}, \quad (11)$$

and Eqs. (6) and (7) become

$$\ddot{u} = 0 \quad ; \quad \ddot{v} = \omega u, \quad (12)$$

where $\omega = \sigma^2 B^2 = 12\pi G B^2 / c^2 > 0$. They are immediately integrated to

$$u(t) = u_1 t + u_2, \quad (13)$$

$$v(t) = \frac{1}{6} u_1 \omega t^3 + \frac{1}{2} u_2 \omega t^2 + v_1 t + v_2, \quad (14)$$

being u_1 , u_2 , v_1 , and v_2 arbitrary integration constants. Taking into account Eq. (5), we find

$$\frac{2}{3} u_1 v_1 - \frac{1}{3} \omega u_2 - \frac{4\pi G}{c^2} D = 0. \quad (15)$$

Since D is a physical parameter, it would be natural to use it as *given* and derive one of the other constants. But this complicates calculations without substantial advantages, as its value depends on the normalization of the scale factor a , which can be fixed arbitrarily. Thus, we will determine D from Eq. (15)

$$D = \frac{c^2}{12\pi G} (2u_1 v_1 - \omega u_2). \quad (16)$$

Being of course $D > 0$, this gives limitations on the choice of u_1 , u_2 , v_1 .

The well known [31,14] solution $\varphi = 2/\sqrt{3(1+\alpha)} \log t$, coming from the potential

$$V(\varphi) = \frac{2(1-\alpha)}{3(1+\alpha)^2} \exp\left(-\sqrt{3(1+\alpha)}\varphi\right), \quad (17)$$

is a very particular case of what we find in Eqs. (13) and (14). It can be obtained by setting

$$B^2 = \frac{2(1-\alpha)}{3(1+\alpha)^2} \quad ; \quad \sigma = \sqrt{3(1+\alpha)} \quad ; \quad u_2 = v_2 = v_1 = 0. \quad (18)$$

Eq. (16) then gives $D = 0$, so that we get a model without matter, not really interesting in our context.

A more interesting possibility is given by the choice $u_1 = v_1 = v_2 = 0$, involving, from Eq. (16), $u_2 < 0$. This implies, in fact,

$$D = -\frac{c^2 \omega u_2}{12\pi G}, \quad a^3 = \frac{1}{2} u_2^2 \omega t^2, \quad \rho_\varphi = \frac{4}{\sigma^2 t^2}. \quad (19)$$

In this particular case, ρ_φ ($\propto a^{-3}$) scales as ρ_m . (But see below for a discussion on the scaling properties.)

Without any special assumptions on constants, we can get many important quantities as functions of u and v (we do not write them explicitly in terms of t , for sake of brevity)

$$\rho_\varphi(u, v) = \frac{1}{\sigma^2} \left(\frac{(\dot{u}v - u\dot{v})^2}{2u^2v^2} + \omega \frac{u}{v} \right), \quad (20)$$

$$p_\varphi(u, v) = \frac{1}{\sigma^2} \left(\frac{(\dot{u}v - u\dot{v})^2}{2u^2v^2} - \omega \frac{u}{v} \right), \quad (21)$$

$$w_\varphi(u, v) = \frac{(\dot{u}v - u\dot{v})^2 - 2\omega u^3v}{(\dot{u}v - u\dot{v})^2 + 2\omega u^3v}, \quad (22)$$

$$H(u, v) = \frac{\dot{u}v + u\dot{v}}{3uv}, \quad (23)$$

$$\Omega_\varphi(u, v) = \frac{(\dot{u}v - u\dot{v})^2 + 2\omega u^3v}{(\dot{u}v + u\dot{v})^2}, \quad (24)$$

$$\Omega_m(u, v) = \frac{24\pi G D u v}{c^2(\dot{u}v + u\dot{v})^2} = \frac{4u\dot{u}v\dot{v} - 2\omega u^3v}{(\dot{u}v + u\dot{v})^2}, \quad (25)$$

the last equality coming from Eq. (16). It can be easily checked that $\Omega_m + \Omega_\varphi = 1$.

B. Physical considerations

We now pass to simplify the situation with a suitable choice of initial conditions. Of course, the following choice is not the only one possible. We shall see, anyhow, that it is able to reproduce observational data. Other more general choices may improve the situation, but we do not treat them here.

If $t_{in} \equiv -u_2/u_1$ (which is always possible, being $u_1 \neq 0$), the scale factor a is zero, and we can show that there is no other time $t > t_{in}$ when this occurs again. We can thus fix the time origin in such a way that $a(0) = 0$. This condition has to be interpreted just as an arbitrary choice of the time origin. The real beginning (of physical meaning) for the model starts a little bit afterwards, at a time t_1 . This delay is otherwise arbitrary, so that this setting does not seem to exclude important cases, as said before, and leads to a great simplification in the formulae. Now, $a(0) = 0$ implies $u_2 = 0$ or $v_2 = 0$, or both. If we

set only one of them to zero, we obtain $\varphi(0) = \infty$ (which could be accepted, but is rather disturbing), and, most of all, $\Omega_\varphi(0) = 1$, which would mean an *initial* scalar-field dominated universe, with a neglectable content of other types of matter. Of course, if we consider the situation in general, the scalar field does dominate. But, as already mentioned, in our case we start after decoupling time, when a matter-dominated behaviour seems to be more natural. So, if we set $u_2 = v_2 = 0$, we get instead (in a matter-dominated situation, then)

$$\varphi(0) = -\frac{1}{\sigma} \log \frac{u_1}{v_1} \quad ; \quad \Omega_\varphi(0) = 0 \quad ; \quad D = \frac{c^2}{6\pi G} u_1 v_1. \quad (26)$$

We prefer to stick to this choice, so that we have

$$u(t) = u_1 t \quad ; \quad v(t) = \frac{1}{6} u_1 \omega t^3 + v_1 t. \quad (27)$$

Let us now define a time scale t_s such that $H(t_s) = 1/t_s$, which is of the order of the age of the universe. This leads to consider a dimensionless time $\tau \equiv t/t_s$. From Eqs. (23) and (27) we get

$$t_s^2 = \frac{6v_1}{\omega u_1}. \quad (28)$$

By means of these choices the formulae found above for the relevant cosmological parameters reduce to

$$\rho_\varphi = \frac{2(3 + 4\tau^2)}{\sigma^2 t_s^2 (1 + \tau^2)^2}, \quad (29)$$

$$p_\varphi = -\frac{2(3 + 2\tau^2)}{\sigma^2 t_s^2 (1 + \tau^2)^2}, \quad (30)$$

$$w_\varphi = -\frac{3 + 2\tau^2}{3 + 4\tau^2}, \quad (31)$$

$$a = (u_1 v_1 t_s^2 (1 + \tau^2) \tau^2)^{1/3}, \quad (32)$$

$$(1 + z)^3 = \frac{\tau_0^2 (1 + \tau_0^2)}{\tau^2 (1 + \tau^2)}, \quad (33)$$

$$H = \frac{2(1 + 2\tau^2)}{3t_s \tau (1 + \tau^2)}, \quad (34)$$

$$\Omega_m = \frac{1 + \tau^2}{(1 + 2\tau^2)^2}, \quad (35)$$

where $z \equiv a(\tau_0)/a(\tau) - 1$ is the redshift, and τ_0 indicates the present time.

If we define dimensionless pressure and energy density

$$\tilde{p}_\varphi \equiv \frac{\sigma^2 t_s^2}{2} p_\varphi \quad ; \quad \tilde{\rho}_\varphi \equiv \frac{\sigma^2 t_s^2}{2} \rho_\varphi, \quad (36)$$

we find the equation of state for the scalar field

$$\tilde{p}_\varphi = \tilde{\rho}_\varphi - 12 + 6\sqrt{4 - \tilde{\rho}_\varphi}, \quad (37)$$

which is well approximated by

$$\tilde{p}_\varphi = -0.382\tilde{\rho}_\varphi - 0.196\tilde{\rho}_\varphi^2. \quad (38)$$

In Fig. 1, we compare the plots of the two functions in Eqs. (37) and (38), and show that the approximation is quite good; a comparison is also made with a straight line $\tilde{p}_\varphi = -0.86\tilde{\rho}_\varphi$, where the coefficient has been obtained through a numerical approximation, as well those in Eq. (38).

From Eqs. (29) and (32) it is possible to derive

$$\delta_1 \equiv \frac{d \log \rho_\varphi}{d \log a} = -\frac{3\tau^2(5 + \tau^2)}{(3 + \tau^2)(1 + 2\tau^2)}. \quad (39)$$

Thus, it is clearly

$$\delta_1 \longrightarrow 0 \quad \text{for } \tau \longrightarrow 0, \quad \delta_1 \longrightarrow -\frac{3}{2} \quad \text{for } \tau \longrightarrow \infty, \quad (40)$$

so that we asymptotically have *two* scaling regimes: $\rho_\varphi \approx \text{const.}$ for early times, and $\rho_\varphi \propto a^{-3/2}$ for late times. In fact, this approximation holds for a very long time, well far from the asymptotic values. Indeed, computing the n -th derivative $\delta_n \equiv a(d\delta_{n-1}/d(\tau^2))/(da/d(\tau^2))$, it is possible to show that they are all asymptotically zero up to, say, $n = 10$.

An estimate of the moment in which the regime changes can be given finding the maximum (or the minimum) of δ_2 . This is achieved at $\tau \simeq 0.3$, and it is remarkable that this result depends only on t_s . All this discussion shows that it is possible to generalise the concept of scaling solutions. The situation is illustrated in Figs. (2) and (3).

As a matter of fact, in the literature it is widely accepted that using an exponential potential leads to a dark energy density which scales like matter. Our results seem to be in contrast with this statement, which is a consequence of assuming w_φ almost perfectly constant. It is not our case, as shown in the following.

As in [6], we can use the function $\Gamma \equiv V''V/(V')^2$. Defining

$$x \equiv \frac{\dot{\varphi}^2}{2V} = \frac{1 + w_\varphi}{1 - w_\varphi}, \quad \dot{x} \equiv \frac{d \log x}{d \log a}, \quad \ddot{x} \equiv \frac{d^2 \log x}{d \log a^2}, \quad (41)$$

it is possible to find [6]

$$\Gamma = 1 + \frac{w_m - w_\varphi}{2(1 + w_\varphi)} - \frac{1 + w_m - 2w_\varphi}{2(1 + w_\varphi)} \frac{\dot{x}}{6 + \dot{x}} - \frac{2}{1 + w_\varphi} \frac{\ddot{x}}{(6 + \dot{x})^2}. \quad (42)$$

If one makes the assumption that w_φ is nearly constant, then $\dot{x} \simeq \ddot{x} \simeq 0$. Since it is $\Gamma = 1$ strictly in the case of our potential, Eq. (42) implies $w_m \approx w_\varphi$. The point is that it is not true that our exact solutions for the exponential potential lead to $\dot{x} \simeq \ddot{x} \simeq 0$. For instance, we have

$$x = \frac{\tau^2}{3(1 + \tau^2)}, \quad \dot{x} = \frac{3}{1 + 2\tau^2}, \quad (43)$$

so that $\dot{x} \longrightarrow 0$ only asymptotically. Since in our model $\tau \lesssim 1$, x and \dot{x} are then far from being zero.

But let us also consider the third term in the right-hand side of Eq. (42) with $w_m = 0$ (matter is simply dust in our model). Substituting our solution, we get

$$-\frac{1+w_m-2w_\varphi}{2(1+w_\varphi)}\frac{\dot{x}}{6+\dot{x}}=\frac{9+8\tau^2}{12\tau^2+16\tau^4}; \quad (44)$$

we see that this expression diverges for $\tau \rightarrow 0$, being always > 0.5 in the useful range of τ .

In our opinion, the main check for the solution in Eq. (27) is thus only its capability to reproduce the experimental results, which we are going to do just below.

From Eq. (35) we get

$$\tau_0^2 = \frac{1-4\Omega_{m0}+\sqrt{1+8\Omega_{m0}}}{8\Omega_{m0}}. \quad (45)$$

Once we give an acceptable value for Ω_{m0} , we obtain a value for τ_0 . For instance, $\Omega_{m0} = 0.3$ gives $\tau_0 = 0.82$, and this implies $w_{\varphi 0} = -0.76$. If the value $H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1}$ is also given, we get

$$t_s = \frac{2(1+2\tau_0^2)}{3H_0\tau_0(1+\tau_0^2)} = \frac{1.14}{H_0}; \quad (46)$$

assuming $h = 0.7$, we have $t_s = 15.8 \times 10^9$ years, and $t_0 = 13 \times 10^9$ years. It is also possible to obtain the relation between $w_{\varphi 0}$ and Ω_{m0}

$$w_{\varphi 0} = \frac{1+8\Omega_{m0}-3\sqrt{1+8\Omega_{m0}}}{4(1-\Omega_{m0})}. \quad (47)$$

For $\Omega_{m0} = 0.2 \div 0.4$, we get $w_{\varphi 0} = -0.699 \div -0.811$, and the value -0.5 is reached only in the case of $\Omega_{m0} = 0$. It is also possible to obtain w_φ as a function of the redshift

$$w_\varphi = -\frac{2\zeta + \sqrt{\zeta(4+\zeta)}}{\zeta + 2\sqrt{\zeta(4+\zeta)}}, \quad (48)$$

where $\zeta \equiv (1+z)^3$.

Figs. 4, 5, and 6 show the dependance of $w - \phi$ versus τ , Ω_{m0} , and ζ , respectively. We can see that w_φ varies much and has values ~ -0.78 . For $\zeta = 1$ (now) it is $w_\varphi = -0.77$, and already for $\zeta = 4$ ($z \simeq 0.59$) we find $w_\varphi \simeq -0.89$. It is remarkable that the values of $|w_\varphi|$ are greater than 0.7. According to our knowledge, this feature is found only in [34].

Another interesting quantity is the present value of φ . After straightforward algebra we get

$$\varphi_0 = -\frac{1}{\sigma} \log \left(\frac{27H_0^2}{16B^2} (1-4\Omega_{m0} + \sqrt{1+8\Omega_{m0}}) \right), \quad (49)$$

and we see that this value depends on the observed parameters and on the value of B^2 , which was until now completely undetermined. Now, for $\tau_0 = 0.82$ and $t_0 = 13 \times 10^9$ years, we have $B^2 = 2.5 \times 10^{-47} \exp(-\sigma\varphi_0) \text{ GeV}^4$. Considering $\varphi_0 \approx 1/6 M_P$ (M_P being the Planck

mass) we see that $\exp(-\sigma\varphi_0) \approx 1$, and we can determine the unknown parameter for the potential

$$V(0) \equiv B^2 \approx 2.5 \times 10^{-47} \text{ GeV}^4. \quad (50)$$

But we have also to observe that a ‘little’ change in φ_0 entails a ‘large’ change in B^2 . For instance, if $\varphi_0 \approx M_P$, then $\exp(-\sigma\varphi_0) \approx 0.0025$ and B^2 changes of three orders of magnitude. Due to Eq. (26), we have that $\varphi(0) = -1/\sigma \log(2\pi G B^2 t_s^2/c^2)$. This means therefore that a relatively wide range of initial values of φ ends up to a narrower set of final φ_0 ’s.

Thus, everything seems to work fine, but things are more complicated. Indeed, one has to ask what is really measured in the supernovae experiment. The value $\Omega_{m0} = 0.3$ is not a direct consequence of the data, since it depends on the model, which uses the constant Λ -term. What we really measure is the distance modulus, so that it is this quantity that we should compare in the two situations. Here, we limit ourselves to a very qualitative discussion.

Let us recall, then, the definitions of luminosity distance (in Mpc)

$$d_L = 3000(1+z) \int_0^z \frac{dz'}{H(z')}, \quad (51)$$

and distance modulus

$$\delta \equiv m - M = 5 \log_{10} d_L(z) + 25. \quad (52)$$

We have thus to compare this last quantity in the case when $H(z)$ is taken from the usual model with Λ [18,21], that is,

$$H(z) = H_0 \sqrt{(1+z)^2(1+\Omega_{m0}z) - z(2+z)(1-\Omega_{m0})}, \quad (53)$$

with the one obtained eliminating τ from Eqs. (33) and (34).

In Fig. 7 we compare δ with $\tilde{\delta}$ (let us mark with a \sim the values for the model with Λ). The agreement is almost perfect, up to 0.06%. But there is a trick! $\tilde{\delta}$ was obtained from a value $\tilde{\Omega}_{m0} = 0.37$. Of course, this value is still in the possible range but at its limit. If we decide to trust strongly on the value $\tilde{\Omega}_{m0} = 0.3$ and want to obtain the same good agreement, we have to change the value of τ_0 to 1.22. This gives a very different value $\Omega_{m0} = 0.16$ in the model with φ . This is again at the limit of possible estimates (due to other investigations on dark matter).

In conclusion, we see that this solution (with the potential in Eq. (8)) is indeed difficult to fully adapt to observed data, but for reasons which are not easy to investigate without general exact solutions. Moreover, it is not clearly incompatible (until we get better data); therefore, it seemed to us useful to present it in detail.

III. TWO EXPONENTIALS COMBINED

A. Mathematical treatment

We now consider a combination of two exponentials, which will give us much better results, as expected. The procedure strictly follows the above one.

Let us consider the potential

$$V(\varphi) = A^2 e^{\sigma\varphi} + B^2 e^{-\sigma\varphi}, \quad (54)$$

with $\sigma^2 = 12\pi G/c^2$ as before, and A^2, B^2 arbitrary parameters. We use, now, the following change of variables

$$a^3 = \frac{u^2 - v^2}{4} \quad ; \quad \varphi = \frac{1}{\sigma} \log \frac{B(u+v)}{A(u-v)}, \quad (55)$$

which is invertible, provided that $a \neq 0$. This leads to

$$V(u, v) = 2AB \frac{u^2 + v^2}{u^2 - v^2}. \quad (56)$$

With these variables Eqs. (6) and (7) are rewritten as

$$\ddot{u} = \omega^2 u \quad ; \quad \ddot{v} = -\omega^2 v, \quad (57)$$

where now

$$\omega^2 = \frac{12\pi GAB}{c^2}. \quad (58)$$

Again, the integration is immediate, and gives the general solutions

$$u(t) = \alpha e^{\omega t} + \beta e^{-\omega t}, \quad (59)$$

$$v(t) = v_1 \sin(\omega t + v_2), \quad (60)$$

with α, β, v_1, v_2 arbitrary constants. As before, we derive D from the constraint in Eq. (5)

$$D = -\frac{c^2 \omega^2 (v_1 + 4\alpha\beta)}{24\pi G}. \quad (61)$$

Being $D > 0$, this implies $v_1 < -4\alpha\beta$. A change in the sign of v_1 has the only effect of changing the sign of φ , and interchanging A^2 with B^2 . So, we can set $v_1 > 0$ without any loss of generality (the case $v_1 = 0$ is obviously equivalent to considering a Λ -term). As a consequence, α and β must be non zero and with opposite signs.

Again, we can write some important functions in terms of u, v

$$\rho_\varphi = \frac{2((\dot{v}u - \dot{u}v)^2 + \omega^2(u^4 - v^4))}{\sigma^2(u^2 - v^2)^2}, \quad (62)$$

$$p_\varphi = \frac{2((\dot{v}u - \dot{u}v)^2 - \omega^2(u^4 - v^4))}{\sigma^2(u^2 - v^2)^2}, \quad (63)$$

$$w_\varphi = \frac{(\dot{v}u - \dot{u}v)^2 - \omega^2(u^4 - v^4)}{(\dot{v}u - \dot{u}v)^2 + \omega^2(u^4 - v^4)}. \quad (64)$$

(This last expression gives $w_\varphi \simeq -1$ when $(\dot{v}u - \dot{u}v)^2 \ll u^4 - v^4$, which certainly can happen for sufficiently large times.)

For the redshift we have

$$(1+z)^3 = \frac{4a_0^3}{u^2 - v^2}, \quad (65)$$

and finally

$$H(u, v) = \frac{2(u\dot{u} - v\dot{v})}{3(u^2 - v^2)}, \quad (66)$$

$$\Omega_m(u, v) = -\frac{(u^2 - v^2)(\dot{v}^2 - \dot{u}^2 + \omega^2(u^2 + v^2))}{(u\dot{u} - v\dot{v})^2}, \quad (67)$$

$$\Omega_\varphi(u, v) = \frac{(\dot{v}u - \dot{u}v)^2 + \omega^2(u^4 - v^4)}{(u\dot{u} - v\dot{v})^2}, \quad (68)$$

with $\Omega_m + \Omega_\varphi = 1$, of course.

B. Physical considerations

The potential in Eq. (54) has a nonzero minimum $V_{min} = A^2 + B^2$. This is unusual in quintessence theory, since $V_{min} = 0$ is optimal to remove the fine-tuning problem. Indeed, $V_{min} \neq 0$ can be seen as a disguised Λ -term. Anyway, in the following we find that this is actually the case when the scalar field is almost stationary near the minimum of the potential. But there is also the possibility of a slower rolling far from the minimum. The situation is then similar to that in Sec. II, but the additional term in the potential now allows to achieve a better agreement with observational data.

Let us now make a trial for the choice of the free parameters. We set again $a(0) = 0$, and ask for nonsingular $\varphi(0)$. The situation, and hence the interpretation, is the same as above. Thus, we pose $\alpha = -\beta = \lambda/2$, $v_2 = 0$. It is also possible to fix an arbitrary normalization for a and set $v_1 = 1$, obtaining at last

$$u(t) = \lambda \sinh(\omega t) \quad ; \quad v(t) = \sin(\omega t). \quad (69)$$

We get now

$$D = \frac{c^2 \omega^2 (\lambda^2 - 1)}{24\pi G}, \quad (70)$$

implying $|\lambda| > 1$, and

$$\varphi(0) = \frac{1}{\sigma} \log \frac{B(\lambda + 1)}{A(\lambda - 1)}. \quad (71)$$

We define a dimensionless time $\tau = \omega t$ and get

$$a(\tau) = \left(\frac{\lambda^2 \sinh^2 \tau - \sin^2 \tau}{4} \right)^{1/3}, \quad (72)$$

$$(1+z)^3 = \frac{\lambda^2 \sinh^2 \tau_0 - \sin^2 \tau_0}{\lambda^2 \sinh^2 \tau - \sin^2 \tau}, \quad (73)$$

$$H(\tau) = \frac{\omega(\sin(2\tau) - \lambda^2 \sinh(2\tau))}{3(\sin^2 \tau - \lambda^2 \sinh^2 \tau)}, \quad (74)$$

$$w_\varphi(\tau) = \frac{\lambda^2(\cosh \tau \sin \tau - \cos \tau \sinh \tau)^2 + (\sin^4 \tau - \lambda^4 \sinh^4 \tau)}{\lambda^2(\cosh \tau \sin \tau - \cos \tau \sinh \tau)^2 - (\sin^4 \tau - \lambda^4 \sinh^4 \tau)}, \quad (75)$$

$$\Omega_m(\tau) = \frac{2(\lambda^2 - 1)(\cos(2\tau) + \lambda^2 \cosh(2\tau) - 1 - \lambda^2)}{(\sin(2\tau) - \lambda^2 \sinh(2\tau))^2}. \quad (76)$$

In comparison with the situation in Sec. II, we now have one more free parameter. This gives the possibility of a much better agreement with observational data.

We can also take into some account the final and initial values of φ , i.e., φ_0 and $\varphi_i = \varphi(0)$. If τ_0 is the present dimensionless time, we get

$$\exp(\sigma(\varphi_i - \varphi_0)) = \frac{\lambda^2 \sinh \tau_0 + \lambda(\sin \tau_0 - \sinh \tau_0) - \sin \tau_0}{\lambda^2 \sinh \tau_0 - \lambda(\sin \tau_0 - \sinh \tau_0) - \sin \tau_0}. \quad (77)$$

When $\lambda \gg 1$, we have $\varphi_i \simeq \varphi_0$. In fact, φ is practically constant and we have $w_\varphi \simeq -1$, with nearly perfect emulation of a cosmological constant. On the other hand, if $\lambda \simeq 1$, then $\exp(\sigma(\varphi_i - \varphi_0)) \simeq 0$. This can be interpreted as $\varphi_i \simeq \infty$, or, better, as the possibility of a wide range of φ_i 's, with nearly the same final φ_0 .

Whatever is λ , anyhow, it is possible to obtain a good agreement with observational data. We here give only two extreme cases, with $\lambda = 30$ and $\lambda = 1.1$.

In the first case (case I) we set

$$\lambda = 30 \quad ; \quad \tau_0 = 1.2 \quad ; \quad \omega = 2.8 \times 10^{-18} s^{-1} = 2.8 \times 10^{-42} GeV, \quad (78)$$

which gives

$$H_0 = 70 \quad ; \quad \Omega_{m0} = 0.3 \quad ; \quad w_{\varphi_0} = -0.999. \quad (79)$$

In the second case (case II) posing

$$\lambda = 1.1 \quad ; \quad \tau_0 = 0.44 \quad ; \quad \omega = 1.07 \times 10^{-18} s^{-1} = 1.07 \times 10^{-42} GeV \quad (80)$$

gives

$$H_0 = 70 \quad ; \quad \Omega_{m0} = 0.3 \quad ; \quad w_{\varphi_0} = -0.76. \quad (81)$$

Again, we find that the value of w_φ is less than -0.7 , as in Sec. II. In Figs. 8 and 9 the distance modulus δ for these two cases is compared with the Λ -term case. As before, the agreement is quite good, but now we have $\tilde{\Omega}_{m0} = 0.30$, so that, with this very rough analysis, it is impossible to make a distinction.

Let us present again the plots of $\log(\rho)$ versus $\log(a)$. They are shown in Figs. 10 and 11. We see that the case I is practically indistinguishable from a Λ -term, while the case II

is more similar to the situation in Sec. II, with different scaling regimes. It is interesting to note that the regimes are now three, even if the last one seems to be important only in the remote future. We stress that also in this case, as well as in Sec. II, w_φ is not constant.

As a final result, we plot in Figs. 12 and 13 the equation of state for the scalar field. Now it is impossible to show an exact analytical expression, so that we only give the plots (in arbitrary units) in the two examined cases. It is interesting to note that in the case I, although so similar to the pure cosmological constant case, we nonetheless obtain a nontrivial plot for the equation of state. But, clearly, this point deserves further investigation.

IV. CONCLUSIONS

We have discussed two particular kinds of potentials which have allowed the general exact integration of Friedmann equations in presence of dust (ordinary and cold dark matter) and scalar field. This has been achieved by performing suitable transformations of variables. Such transformations have not been guessed by chance, but are the results of a well known procedure, the *Nöther Symmetry approach* [43,44,47,42], based on an action principle. This was not mentioned before because it was unnecessary to the main goal of the discussion. Nevertheless, it now seems appropriate to stress the power of such a procedure, which allows to solve cosmological equations, often giving also informations on the potential and/or the possible coupling between the scalar field and the curvature of spacetime, without any limitation on the validity of the solution itself. (For details, see the literature quoted above, and the references therein.)

We have seen that, with a suitable choice of integration constants for both potentials, it is possible to reproduce the main recent results from supernovae (initially interpreted in a Λ -model), with considerable precision, especially in the case II of the second potential. (The case I does not add much to what is already known in a constant Λ -term model.) What is interesting, in our opinion, is that such kinds of models can bring to a different evaluation of an important quantity like Ω_{m0} . This, in a certain sense, sheds new light on the exponential potential of the first type, which is usually not considered as completely adequate for quintessence, for instance. But we have seen that, without considering *a priori* w_φ as a constant and having a general exact solution, something else can be learned. As a matter of fact, not having an almost constant w_φ makes it impossible to treat the tracker condition in the usual way. Also, we get appreciable values of w_φ , i.e., surely less than -0.7 .

Of course, all our discussion is in part still *qualitative*, in that we should need to make a more punctual analysis of observational data, and verify the best fit with the various models, in order to see whether and when real differences arise. Anyway, our analysis already seems to confirm some of the considerations made in [48].

Another important point to note is that, to be realistic and cover the whole (or, at least, a substantially wider) range of the life of the universe, radiation (and hot dark matter) must be added into the game. This could allow to study the CMBR spectrum and the formation of structures. But it presumably destroys the possibility to integrate the system of the cosmological equations, leading to the necessity of using the results we established here only as a guide for a more complete analysis.

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Figure Captions

Fig. 1 - Comparison of the state equation in Eq. (36) with an approximate quadratic equation and with an approximate straight line.

Fig. 2 - The straight line represents the dependence of ρ_m versus a ($\rho_m = Da^{-3}$). The thick curve is for the scalar field. The two tangents show the asymptotic scaling behaviour in the period of dominating matter and scalar field, respectively. The bullets indicate present time, according to $a = 0.82$. Units are arbitrary.

Fig. 3 - The derivative of $\log(\rho)$ with respect to $\log(a)$, shows a quick transition from one scaling regime to another one. Units are arbitrary.

Fig. 4 - The plot of w_ϕ versus time shows that w_ϕ is far from being constant.

Fig. 5 - $w_{\phi 0}$ versus Ω_{m0} .

Fig. 6 - This plot shows the dependence of w_ϕ from the redshift.

Fig. 7 - Comparison of the distance modulus δ (derived from Eqs. (32), (33) and (34), and assuming $\Omega_{m0} = 0.3$) (continuous line) with $\tilde{\delta}$ (derived from a Λ -term model, assuming a $\tilde{\Omega}_{m0} = 0.37$) (dots).

Fig.8 - Comparison of the distance modulus δ] (derived from Eqs. (68), (69) and (71) in case I, and assuming $\Omega_{m0} = 0.3$) (continuous line) with $\tilde{\delta}$ (derived from a Λ -term model, assuming $\tilde{\Omega}_{m0} = 0.3$)) [dots].

Fig.9 - Comparison of the distance modulus δ (derived from Eqs. (68), (69) and (71) in case II, and assuming $\Omega_{m0} = 0.3$) (continuous line) with a (derived from a Λ -term model, assuming $\tilde{\Omega}_{m0} = 0.3$)) [dots].

Fig. 10 - Case I. The straight line with slope represents the dependance of $\log \rho_m$ versus $\log a$ ($\rho_m = Da^{-3}$). The horizontal thick line is for the scalar field. It emulates a constant. The bullet indicates present time, according to $\tau_0 = 1.2$. Units are arbitrary.

Fig. 11 - Case II. The straight line with slope represents the dependance of $\log \rho_m$ versus $\log a$ ($\rho_m = Da^{-3}$). The thick line is for the scalar field. There are three approximate scaling regimes. The bullet indicates present time, according to $\tau_0 == 0.44$. Units are arbitrary.

Fig.12 - Case I. State equation for the scalar field. Although it perfectly emulates a cosmological constant, the equation is nontrivial. Units are arbitrary, but coherent with Fig. 13.

Fig.13 - Case II . State equation for the scalar field. Units are arbitrary, but coherent with Fig. 12.

Fig. 1

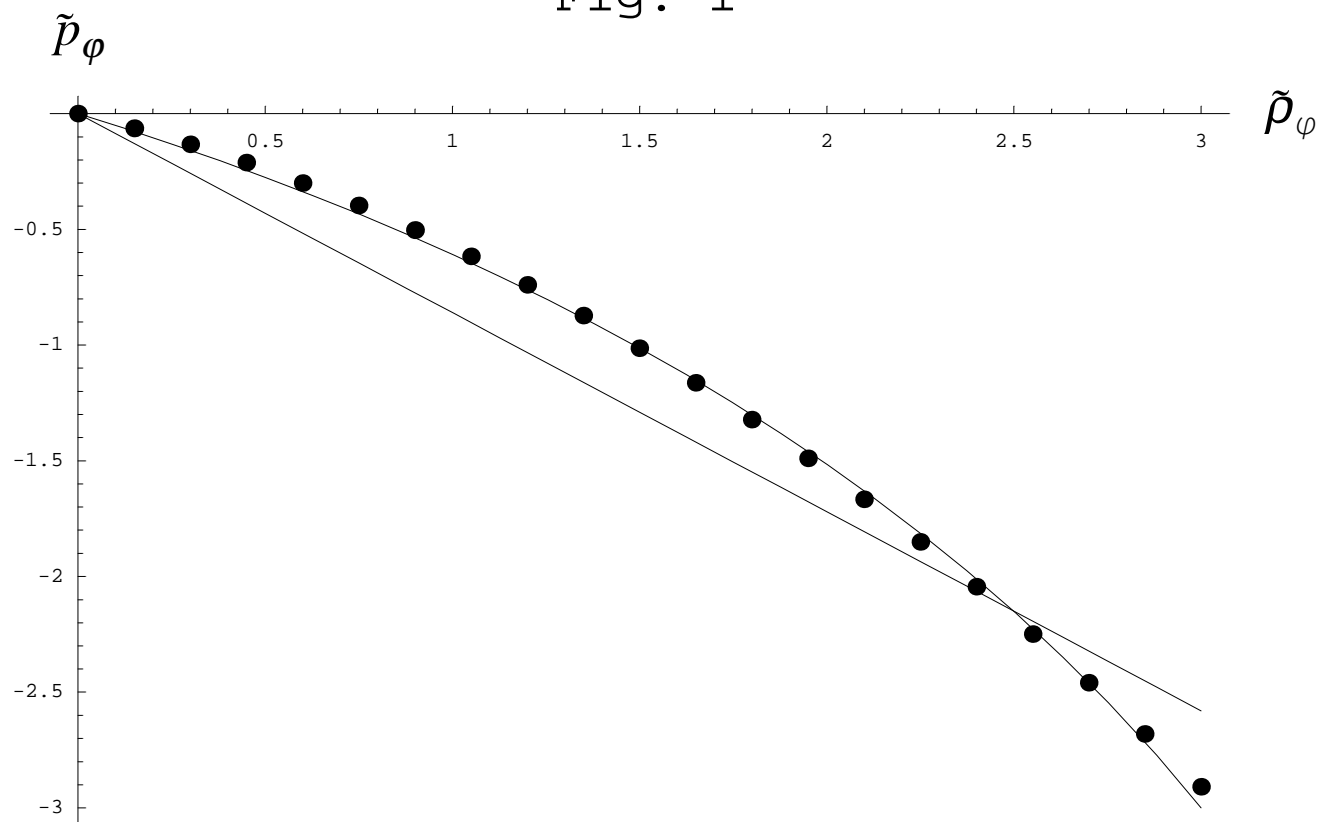


Fig. 1 - Comparison of the state equation in Eq. (36) with an approximate quadratic equation and with an approximate straight line.

Fig. 2

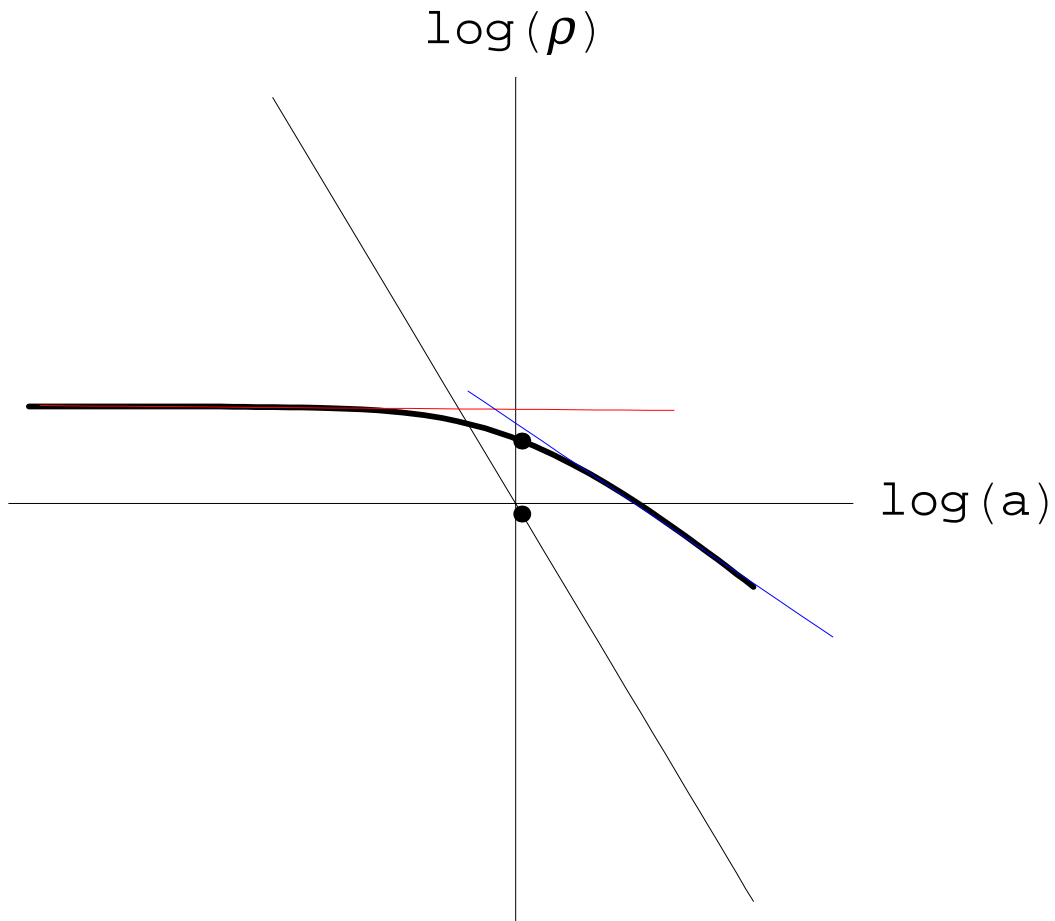


Fig. 2 - The straight line represents the dependence of ρ_m versus a ($\rho_m = D a^{-3}$). The thick curve is for the scalar field. The two tangents show the asymptotic scaling behaviour in the period of dominating matter and scalar field, respectively. The bullets indicate present time, according to $\tau_0 = 0.82$. Units are arbitrary.

Fig. 3

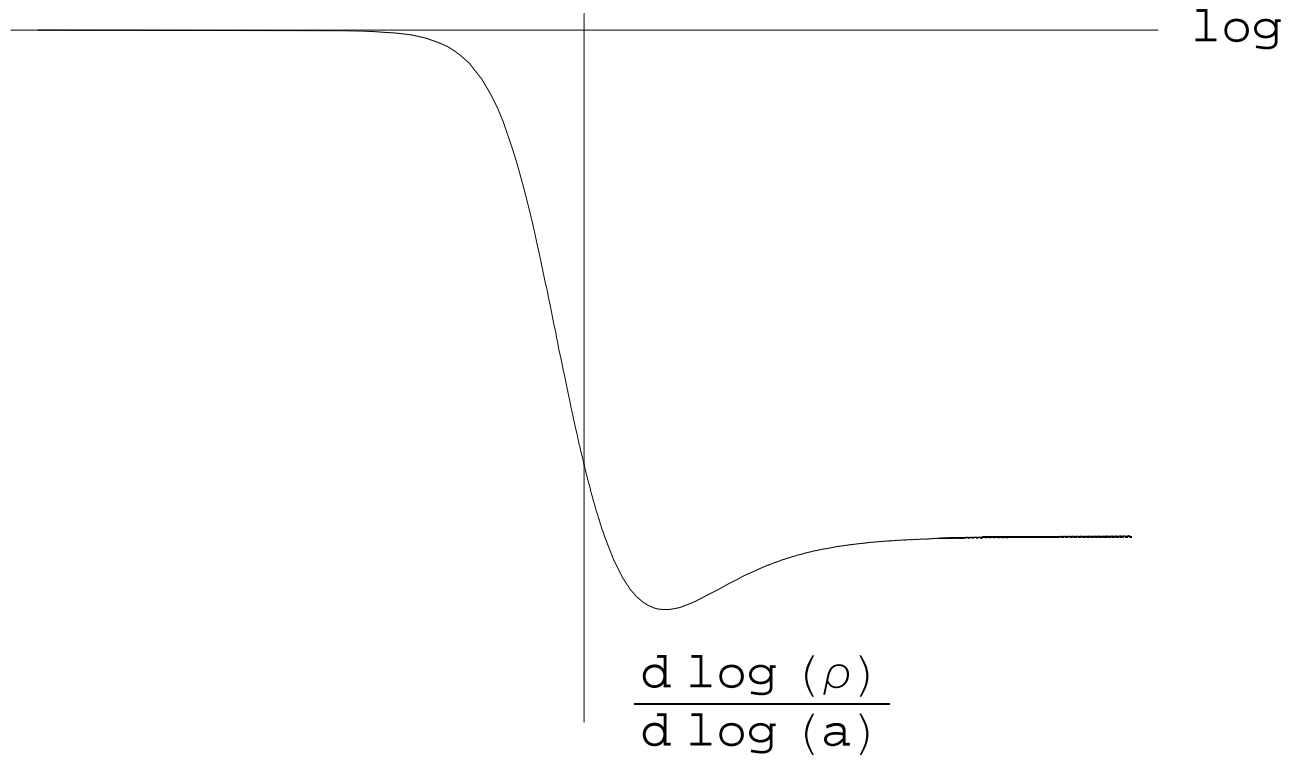


Fig. 3 - The derivative of $\log(\rho)$ w.r.t. $\log(a)$, shows a quick transition from one scaling regime to another one. Units are arbitrary.

Fig. 4

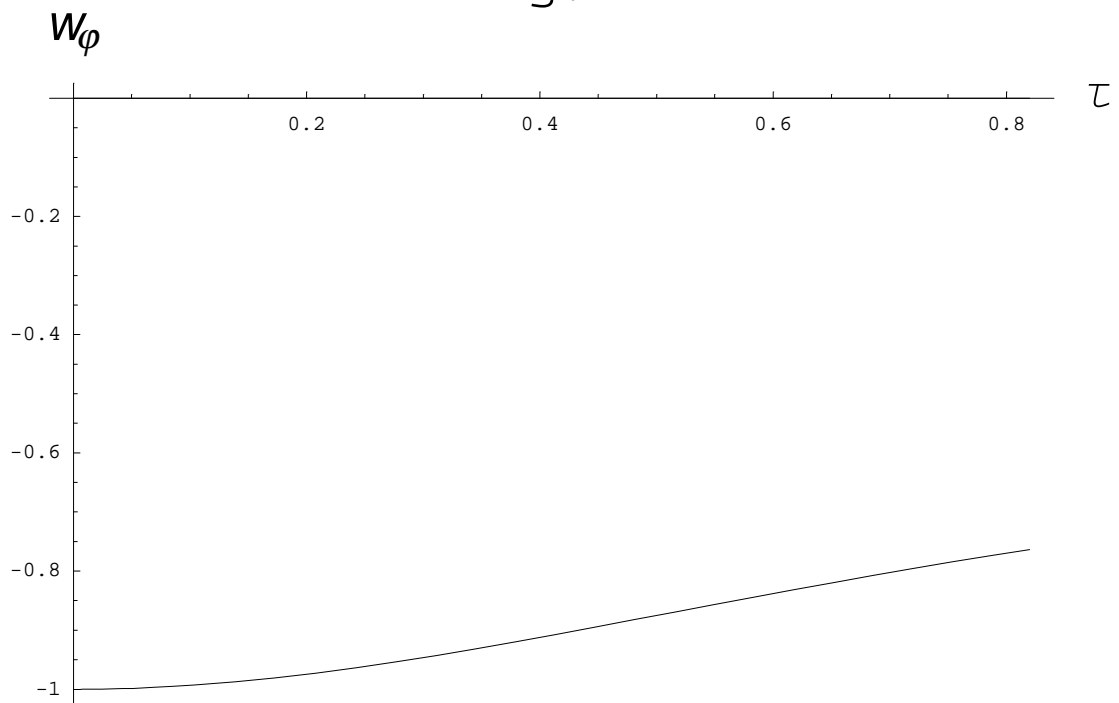


Fig. 4 - The plot of w_ϕ versus time shows that w_ϕ is far from being constant.

Fig. 5

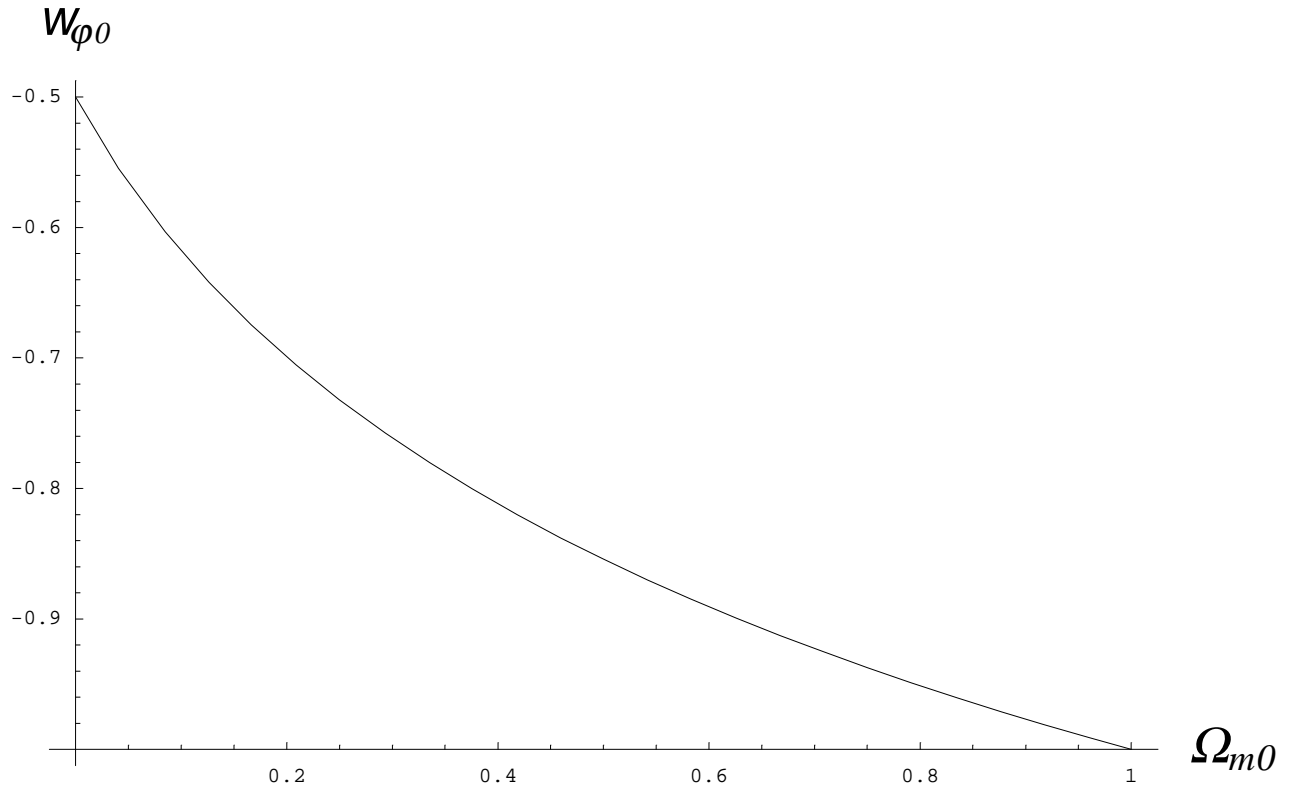


Fig. 5 - $w_{\varphi 0}$ versus Ω_{m0} .

Fig. 6

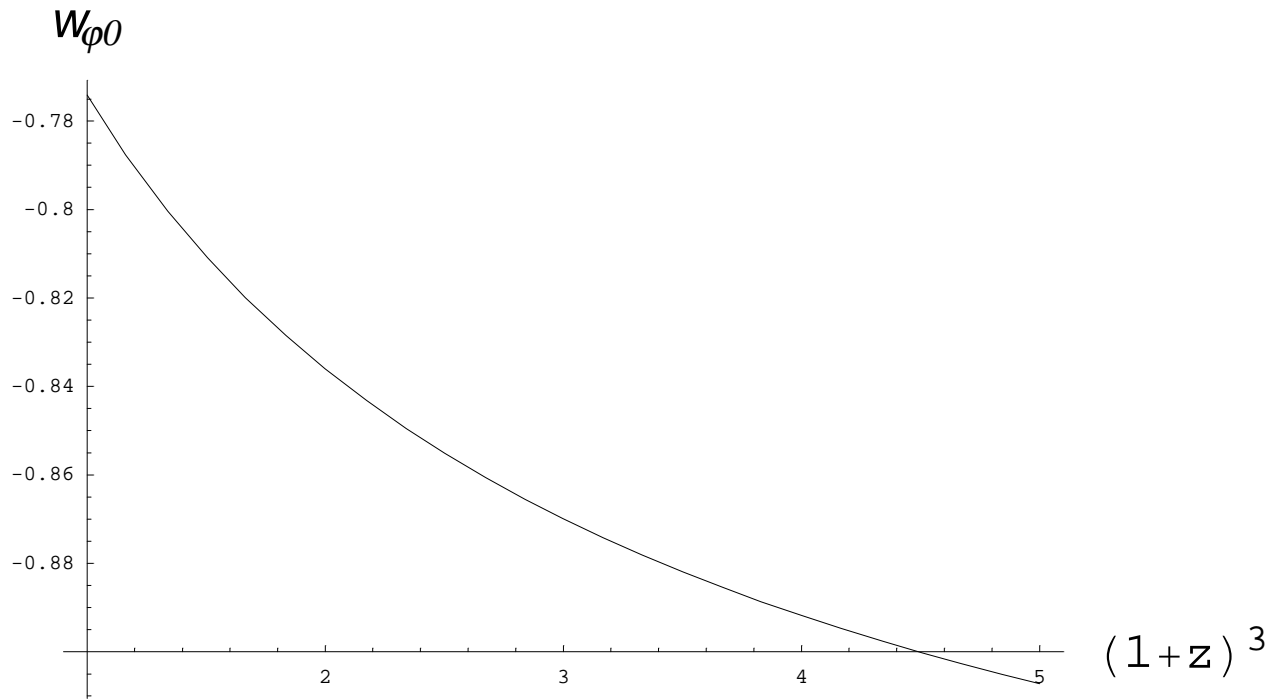


Fig. 6 - This plot shows the dependence of w_{φ} from the redshift.

Fig. 7

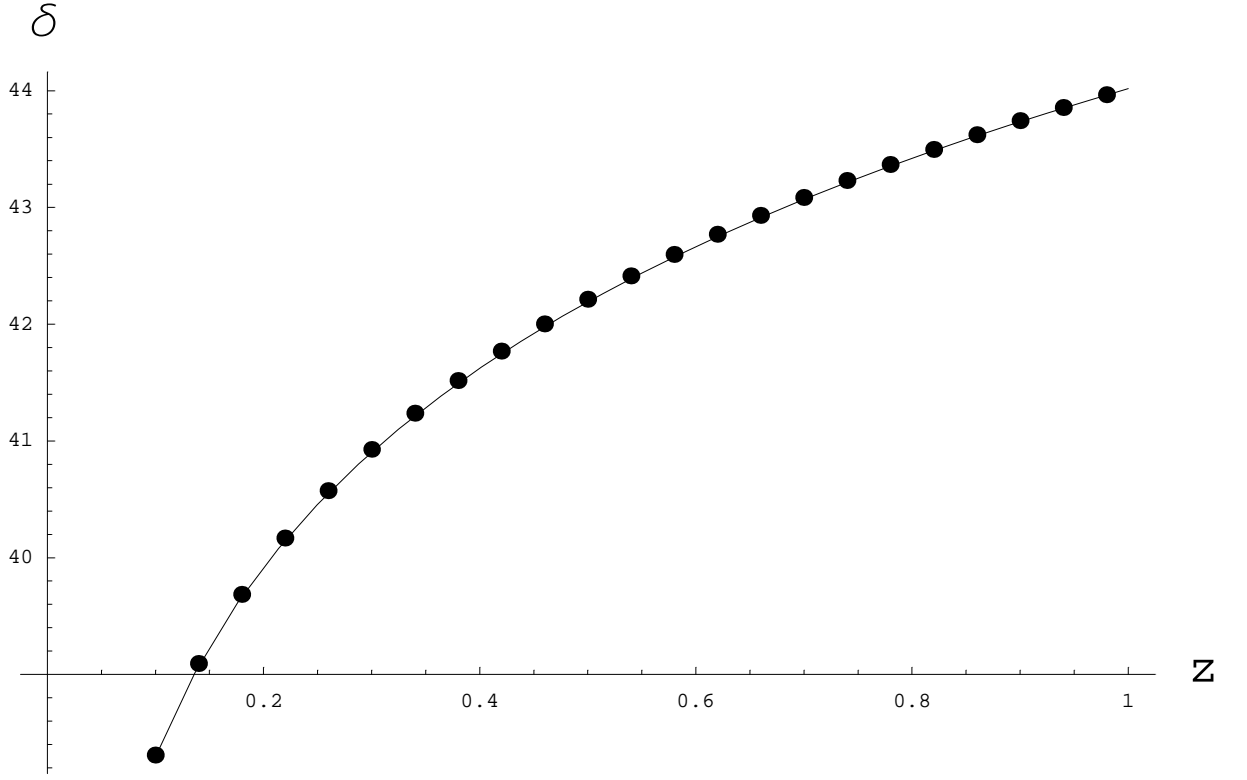


Fig. 7 - Comparison of the distance modulus δ (derived from Eqs. (32), (33) and (34), and assuming $\Omega_{m0} = 0.3$) [continuous line] with $\tilde{\delta}$ (derived from a Λ -term model, assuming $\tilde{\Omega}_{m0} = 0.37$) [dots].

Fig. 8

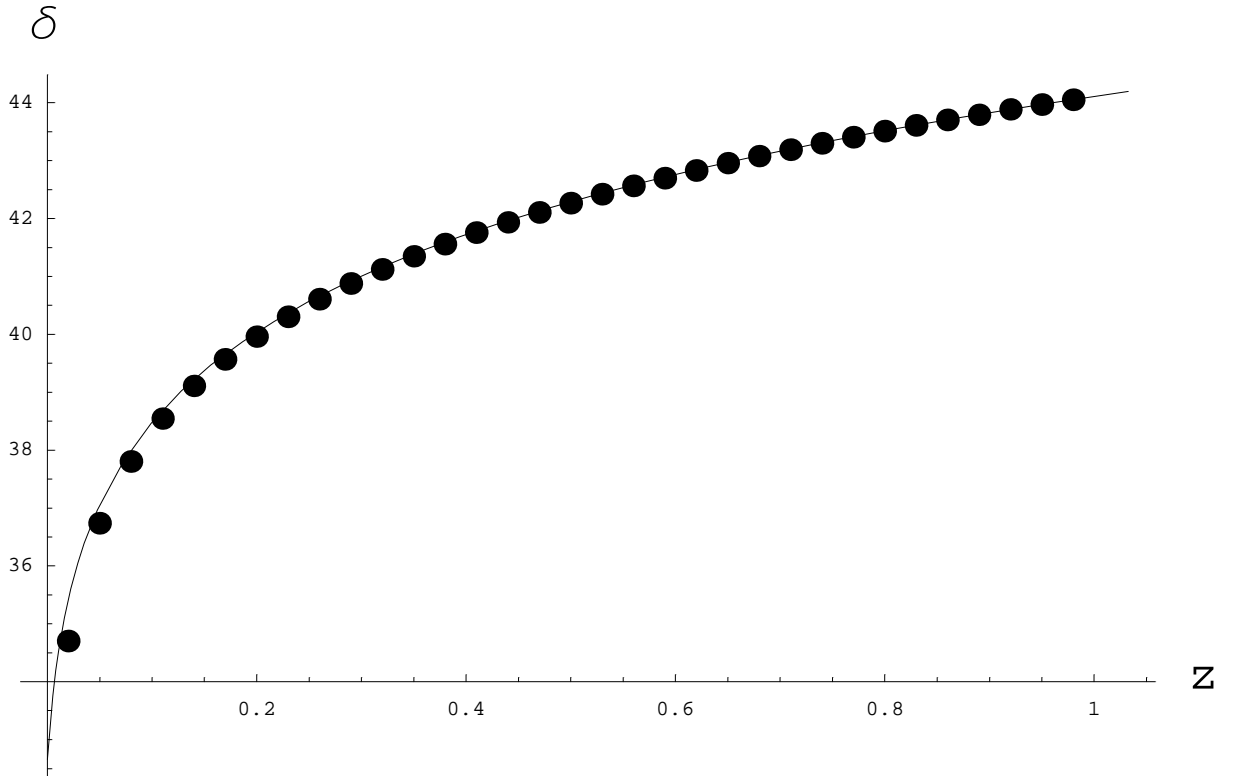


Fig.8 - Comparison of the distance modulus δ (derived from Eqs. (68), (69) and (71) in case I, and assuming $\Omega_{m0} = 0.3$) [continuous line] with $\tilde{\delta}$ (derived from a Λ -term model, assuming $\tilde{\Omega}_{m0} = 0.3$) [dots].

Fig. 9

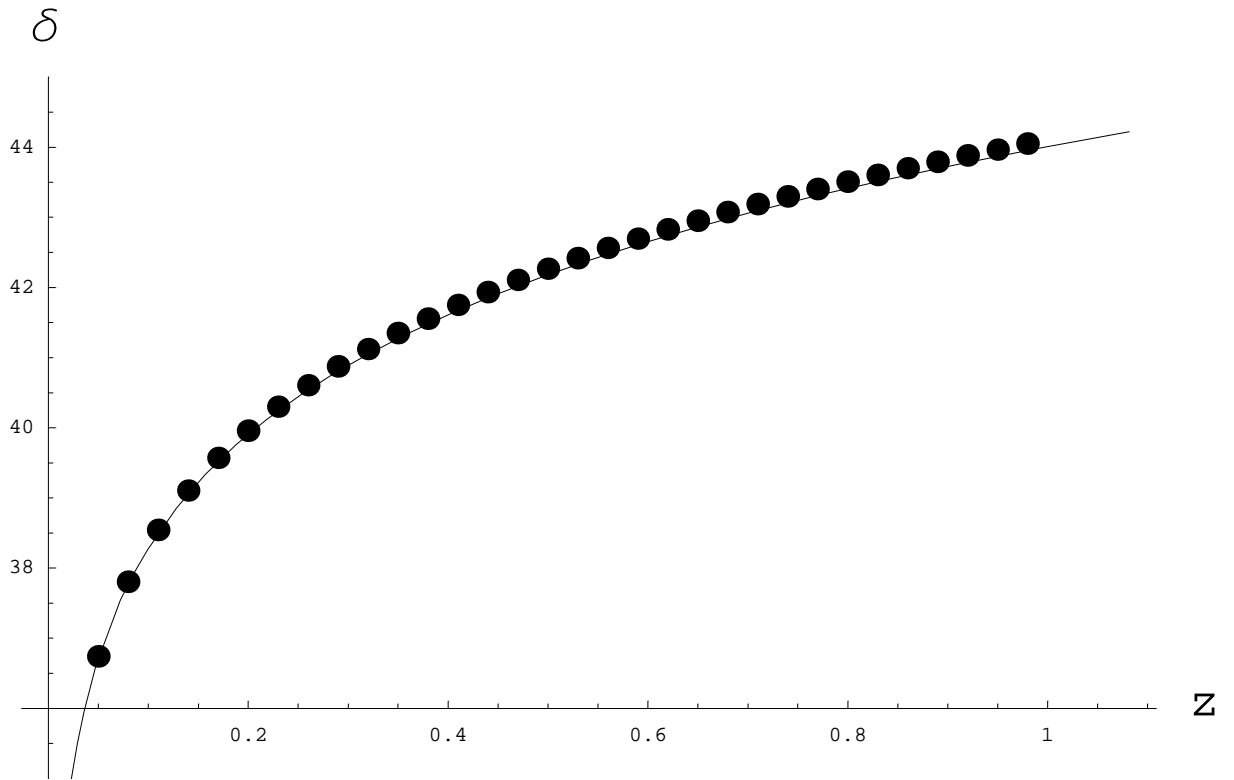


Fig.9 - Comparison of the distance modulus δ (derived from Eqs. (68), (69) and (71) in case II, and assuming $\Omega_{m0} = 0.3$) [continuous line] with $\tilde{\delta}$ (derived from a Λ -term model, assuming $\tilde{\Omega}_{m0} = 0.3$) [dots].

Fig. 10

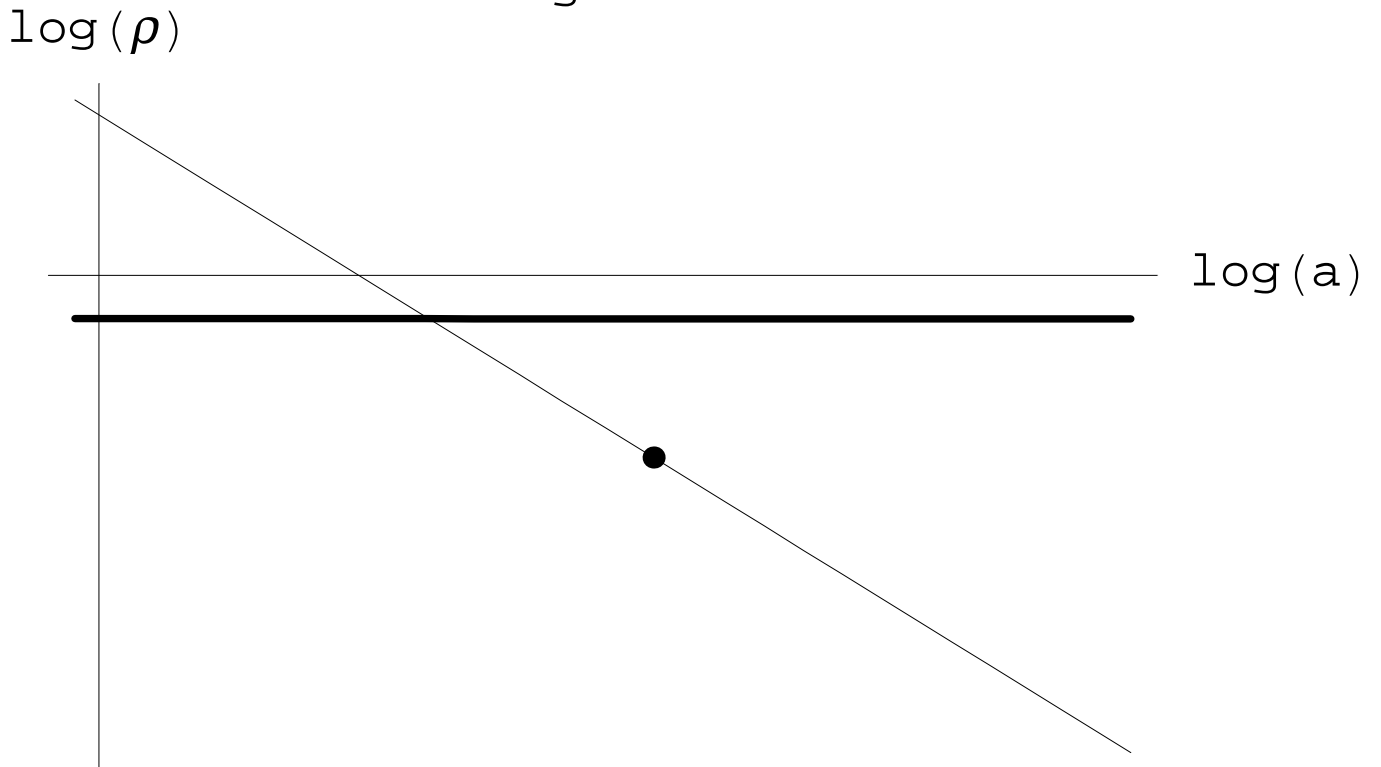


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Fig. 11

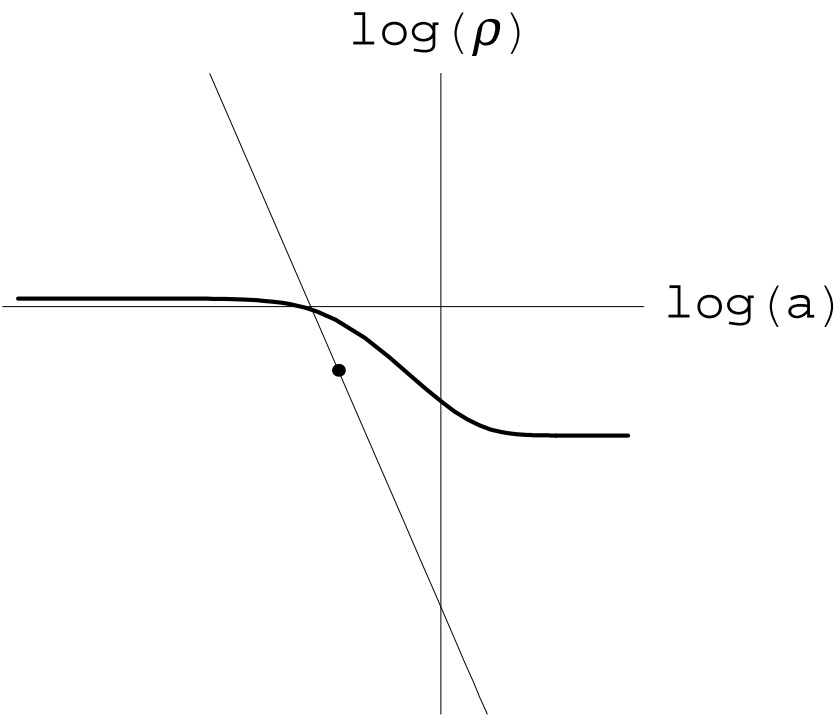


Fig. 11 - Case II. The straight line with slope represents the dependance of ρ versus a ($\rho = D a^{-3}$). The thick line is for the scalar field. There are three approximate scaling régimes. The bullet indicates present time, according to $\tau_0 = 0.44$. Units are arbitrary.

Fig. 12

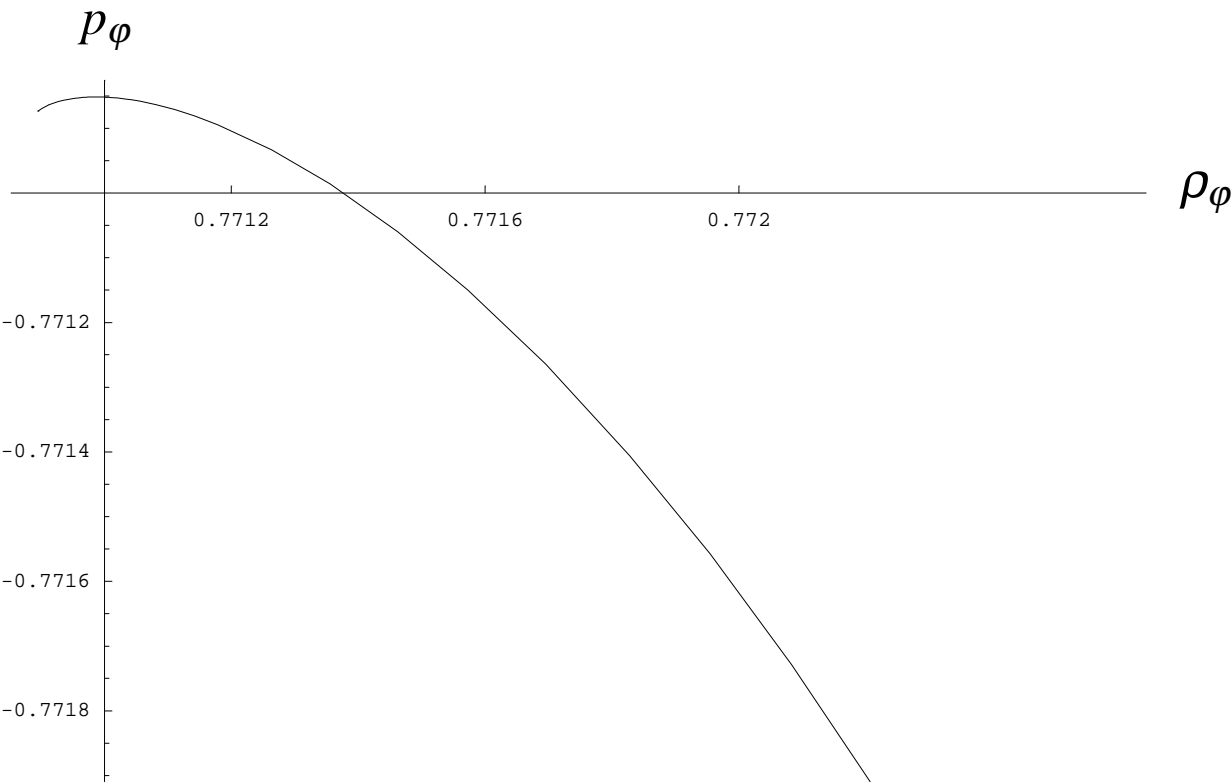


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Fig. 13

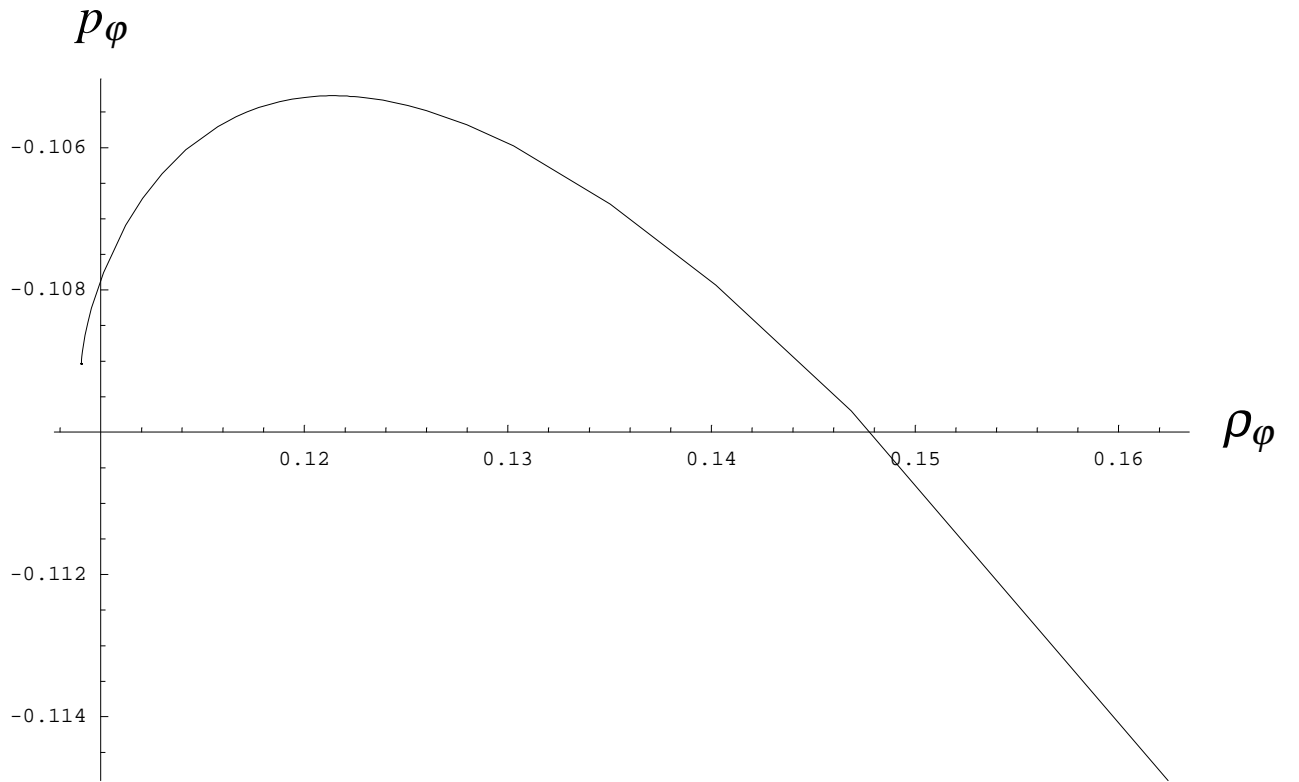


Fig. 13 – Case II. State equation for the scalar field. Units are arbitrary, but coherent with Fig. 12.